

Statistical Entropy of Near-Extremal and Fundamental Black p -Branes

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Abstract:

The problem of asymptotic density of quantum states of fundamental extended objects is revised in detail. We argue that in the near-extremal regime the fundamental p -brane approach can yield a microscopic interpretation of the black hole entropy. The asymptotic behavior of partition functions, associated with the p -branes, and the near-extremal entropy of five-dimensional black holes are explicitly calculated.

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1 Introduction

Semi-classical methods have revealed many intriguing thermodynamic properties of black holes. In recent studies these methods have been applied to the calculation of thermodynamic quantities for black hole and extended objects in string theory or M -theory. As a result the microscopic description of entropy in terms of the counting of string states [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] have been obtained. It has been shown that the dilatonic scalar fields of the p -brane soliton have to be finite on the horizon in the near-extremal limit in order for the tree-level approximation to be reliable.

Note that isotropic extremal p -brane solutions of this type are very limited. All of these cases were enumerated in Ref. [15]. Let us start with brief remarks on this list.

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The elementary membrane [16] and solitonic 5-brane in $D = 11$ [17]. For $D = 11$ there is no dilaton field, and the microscopic interpretation of the entropy of the membrane and 5-brane was presented in Ref. [14].

The self-dual 3-brane in $D = 10$ [18, 19]. For $D = 10$ the dilaton field decouples from the self-dual 3-brane; the microscopic discussion of the entropy can be found in Ref. [7].

In the remaining cases the dilatonic scalar fields do not decouple for generic values of charges, but they do remain finite at the horizon in the extremal limit. When all charges are equal, the dilatonic scalar fields then decouple; the resulting solutions are non-dilatonic p -branes [14]. In this case the dyonic string becomes the self-dual string in $D = 6$ [20].

Black holes in $D = 5$ and $D = 4$ with three and four independent participating field strengths respectively [21]. In these cases the black holes become the Reissner-Nordström black holes [22]. For $D = 5$ black hole solution is stainless [23], but can be oxidised to a boosted dyonic string in $D = 6$ dimensions. It was shown that the entropy per unit p -volume is preserved under the oxidation and can be associated with the microscopic counting of the corresponding D -string states in the near-extremal regime [1].

In $D = 4$ there are two regular black holes. One of them involves four field strengths, and reduces to the Reissner-Nordström black hole when all the charges are equal [10, 11]. This is the case when multiply-charged black holes (see Ref. [24]) may be regarded as bound states at threshold of singly-charged black holes [25, 26, 27, 28, 29]. The other involves only two field strengths, each with electric and magnetic charges. This solution was referred to as the dyonic black hole of the second type in [30]. The multi-center solutions of charged, dilatonic and non-extremal black holes in $D = 4$ can be found also in [31].

In all these cases the number N of non-vanishing charges compatible with having some preserved supersymmetry has to be maximal. The non-extremal generalization of these solutions were presented in Ref. [32]. Recently the cosmological solutions in string and M -theory have been analysed in [33, 34].

There are many more p -brane solitons in the string or M -theory in addition to the cases mentioned above. But at the classical level the ideal-gas relation between entropy and temperature breaks down in all these other cases. It happens due to the fact that for all these solitons the dilaton, as well as the curvature and field strengths diverges on the horizon in the near-extremal limit. The divergences may indicate a breakdown of validity of the classical approximation. The inclusion of string and worldsheet loop corrections can remove such singularities [15].

It has been shown that in the non-dilatonic cases, the entropy and temperature satisfy the massless ideal-gas relation $S \sim T^p$ [14], which is in concordance with the microscopic D -brane picture. This analysis can be extended to the regular dilatonic examples [15].

The purpose of the present paper is to resolve these problems, comparing the statistical properties of quantum states of near-extremal black and fundamental p -branes.

The contents of the paper are the following. In Section 2 we summarize a description of scalar p -brane classes in various dimensions. The statistical entropy associated with some interacting fundamental p -brane excitation modes and its comparison to the Bekenstein-Hawking and near-extremal black p -branes entropy is given in Section 3. In Section 4 the explicit form of quantum counting string states and the entropy of five-branes is computed. Finally the Appendices contain explicit results for fundamental (super) p -branes, namely: mass operators (Appendix A), asymptotic expansions of generating functions (Appendix B), and the one-loop free energy (Appendix C).

2 Classes of Scalar p -Branes

In this section we start with the properties of classical p -branes. Let us consider the relevant part of the tree-level approximation of the string effective action. The bosonic Lagrangian has

the form [21, 30, 32]

$$e^{-1}\mathcal{L} = R - \frac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{2n!} \sum_{\alpha=1}^N e^{-\vec{a}_\alpha \cdot \vec{\phi}} (F^\alpha)^2, \quad (2.1)$$

where $\vec{\phi} = (\phi_1, \dots, \phi_N)$ is a set of N scalar fields, and F^α is a set of N antisymmetric field strengths of rank n , which give rise to a p -brane with world volume dimension $d = n - 1$ if they carry electric charges, or with $d = D - n - 1$ if they carry magnetic charges. The constant (dilaton) vectors \vec{a}_α are vectors characteristic of the supergravity theory associated with low energy limit of string or M - theory.

The usual form of the metric is given by

$$ds^2 = e^{2A(r)} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)} dy^m dy^m, \quad (2.2)$$

where $x^\mu (\mu = 0, \dots, d-1)$ and y^m are the coordinates of the $(d-1)$ - dimensional brane volume and $(D-d)$ -dimensional transverse space respectively. The functions $A(r)$, $B(r)$ and the dilaton scalar $\vec{\phi}$ depend on $r = (y^m y^m)^{1/2}$ only. Note that the metric ansatz therefore preserves an $SO(1, d-1) \otimes SO(D-d)$ subgroup of the original $SO(1, D-1)$ Lorentz group.

For each n -index ($n \geq 2$) field strengths F^α there are two different equations that also preserve the same subgroup [35, 36, 37]:

$$F_{m\mu_1 \dots \mu_{n-1}}^\alpha = \epsilon_{\mu_1 \dots \mu_{n-1}} \left(e^{C_\alpha(r)} \right)' \frac{y^m}{r}, \quad (2.3)$$

$$F_{m_1 \dots m_n}^\alpha = \lambda_\alpha \epsilon_{m_1 \dots m_n p} \frac{y^p}{r^{n+1}}. \quad (2.4)$$

Here $C_\alpha(r)$ is a some differentiable function, ϵ_μ is the volume form on the unit sphere whose metric is $d\Omega^2$, and a prime denotes a derivative with respect to r . The Eq. (2.3) gives rise to an elementary $(d-1)$ -dimensional brane with $d = n-1$, $n = 2, 3, 4$ and electric charges λ_α . While for a solitonic $(d-1)$ -dimensional brane with $d = D-n-1$, $n = 1, 2, 3, 4$ and magnetic charges λ_α the second Eq. (2.4) holds.

It should be noted that for $D = 2n$ some field strengths F^α might be the duals of the original ones of the same degree. Such a particularly interesting class of solutions having both elementary and solitonic contributions refers as dyonic solutions of the first type [30] which are possible only for $n = 2$ ($D = 4$).

If the dot products $\mathcal{M}_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta$ satisfy

$$\mathcal{M}_{\alpha\beta} = 4\delta_{\alpha\beta} - \frac{2d\tilde{d}}{2(D-2)}, \quad (2.5)$$

where $\tilde{d} = D-d-2$, then the Lagrangian (2.1) can be embedded into the string or M -theory [21].

For the black p -branes with N non-vanishing charges the metric has the form [32]

$$ds^2 = e^{2A(r)} \left(e^{-2f} dt^2 + dx^i dx^i \right) + e^{2B(r)} \left(e^{-2f} dr^2 + r^2 d\Omega^2 \right), \quad (2.6)$$

where

$$\exp(2A(r)) = \prod_{\alpha=1}^N \left(1 + \frac{k}{r^{\tilde{d}}} \sinh^2 \mu_\alpha \right)^{-\tilde{d}/(D-2)}, \quad (2.7)$$

$$\exp(2B(r)) = \prod_{\alpha=1}^N \left(1 + \frac{k}{r^{\tilde{d}}} \sinh^2 \mu_\alpha \right)^{d/(D-2)}. \quad (2.8)$$

The d -dimensional world volume of the p -brane is parametrised by the coordinates (t, x^i) , while the remaining coordinates are r and the coordinates on the $(D-d-1)$ -dimensional unit sphere.

It can be shown that the function $f(r)$ has a completely universal form [32], namely $\exp(2f(r)) = 1 - kr^{-\tilde{d}}$. The dilatonic scalar fields $\varphi_\alpha = \vec{a}_\alpha \cdot \vec{\phi}$ can be given by

$$\exp\left(-\frac{1}{2}\epsilon\varphi_\alpha\right) = \left(1 + \frac{k}{r^{\tilde{d}}} \sinh^2 \mu_\alpha\right) \prod_{\beta=1}^N \left(1 + \frac{k}{r^{\tilde{d}}} \sinh^2 \mu_\beta\right)^{-\frac{d\tilde{d}}{2(D-2)}}, \quad (2.9)$$

where $\epsilon = 1$ (-1) for the elementary (solitonic) solutions. The metric of Eqs. (2.6)-(2.8) has a curvature singularity at $r = r_- = 0$ and an outer horizon at $r = r_+ \equiv k^{1/\tilde{d}}$. Generally speaking even at the origin in the extremal limit $r_+ \mapsto r_- = 0$ the curvature remains singular. Furthermore the charges for each field strength are given by $\lambda_\alpha = \frac{1}{2}\tilde{d}k \sinh 2\mu_\alpha$. In the extremal limit we should take $k \mapsto 0$, $\mu_\infty \mapsto \infty$ and keep the charges λ_α fixed.

When all charges are equal the Lagrangian (2.1) reduces to the functional which describes a single scalar p -brane and a single field strength. In general case for the non-singular matrix $\mathcal{M}_{\alpha\beta}$ the constant a , field ϕ and strength F are given by [30]

$$a^2 = \left(\sum_{\alpha,\beta} (\mathcal{M}^{-1})_{\alpha\beta} \right)^{-1}, \quad (2.10)$$

$$\phi = a \sum_{\alpha,\beta} (\mathcal{M}^{-1})_{\alpha\beta} \vec{a}_\alpha \cdot \vec{\phi}, \quad (2.11)$$

$$(F^\alpha)^2 = a^2 \sum_{\beta} (\mathcal{M}^{-1})_{\beta}^\alpha F^2. \quad (2.12)$$

The dilaton prefactor a can be parametrised by $a^2 = \Delta - 2d\tilde{d}(D-2)$. Supersymmetric solutions are associated to $\Delta = 4/N$, with N field strengths participating in the p -brane solution. In addition the functions $A(r)$ and $B(r)$ can be written as follows [32]:

$$\exp(2A(r)) = \left(1 + \frac{k}{r^{\tilde{d}}} \sinh^2 \mu\right)^{-\frac{4\tilde{d}}{\Delta(D-2)}}, \quad (2.13)$$

$$\exp(2B(r)) = \exp\left(-2A(r)\frac{d}{\tilde{d}}\right). \quad (2.14)$$

In the case of $\Delta = 4$, pure electric or pure magnetic black p -branes have been considered in Ref. [18] for $D = 10$ dimensions and in Ref. [38] for $D \leq 11$ dimensions. The analysis was generalized to other values of Δ in the case of near-extremal p -branes in Refs. [23, 30].

3 Entropy of Near-Extremal Black and Fundamental p -Branes

We first begin with the discussion of the thermodynamic properties of classical p -branes. The Hawking temperature and the entropy per unit p -volume of the black p -brane are given by [32]

$$T = \frac{\tilde{d}}{4\pi r_+} \prod_{\alpha=1}^N (\cosh \mu_\alpha)^{-1}, \quad (3.1)$$

$$S = \frac{1}{4} r_+^{\tilde{d}+1} V(S^{\tilde{d}+1}) \prod_{\alpha=1}^N \cosh \mu_\alpha, \quad (3.2)$$

where $V(S^{\tilde{d}+1}) = 2\pi^{\tilde{d}/2+1} \left[(\frac{1}{2}\tilde{d})! \right]^{-1}$ is the volume of the unit $(\tilde{d}+1)$ -dimensional sphere. In the near-extremal limit, i.e. $k \ll \lambda_\alpha$ for all α the relation between entropy and temperature takes the form

$$S \approx \frac{\tilde{d}}{16\pi} V(S^{\tilde{d}+1}) \left(16\pi^2 \tilde{d}^{-N-2} \right)^{\frac{\tilde{d}}{N\tilde{d}-2}} \left(\prod_{\alpha=1}^N \lambda_\alpha \right)^{\frac{\tilde{d}}{N\tilde{d}-2}} T^{\frac{2(\tilde{d}+1)-N\tilde{d}}{N\tilde{d}-2}}. \quad (3.3)$$

If

$$N = \frac{2(D-2)}{d\tilde{d}}, \quad (3.4)$$

then the entropy and temperature are related as follows

$$S \approx \frac{\tilde{d}}{16\pi} V(S^{\tilde{d}+1}) \left(16\pi^2 \tilde{d}^{-N-2} \right)^{\frac{d}{2}} \left(\prod_{\alpha=1}^N \lambda_\alpha \right)^{\frac{d}{2}} T^{d-1} \sim T^p. \quad (3.5)$$

This dependence looks like the natural entropy of massless ideal gas predicted by D -brane considerations. Indeed, open strings on a Dirichlet p -branes can be analyzed as an ideal gas of massless objects in a p -dimensional space. For the equal charge parameters λ_α the dilatonic scalars decouple and the Eq. (3.5) reduces to the relation that is consistent with the one found in Ref. [14]. The above relation holds even if the charges are not equal (the dilatonic scalar fields do not decouple but remains finite at the horizon in the extremal limit) [15]. Note also that thermodynamic properties of black p -branes have been recently discussed in ref.[61].

Note that in the near-extremal limit the curvature and the field strengths are also finite at the horizon when the condition (3.4) is satisfied [15]. For other values of N the relation $S \sim T^{d-1}$ in the near-extremal limit breaks down. It can be expected since dilatonic scalar fields, the field strengths and the curvature diverge at the horizon in this case. In fact the tree-level approximation is not sufficient for dilatonic p -branes. But for loop effects, the dilaton fields, field strengths and curvature may be finite at the horizon and, therefore, the relation between entropy and temperature of the quantum p -branes will satisfy precisely the natural ideal-gas scaling [14, 15].

In the near-extremal regime for all p -brane solitons satisfying the condition (3.4) the temperature goes to zero. In this situation the relation between the entropy and mass of the near-extremal p -branes can also be written as

$$S \sim (\delta M^2)^{\frac{d-1}{d}}. \quad (3.6)$$

On the other hand adapting to the thermodynamics of fundamental p -branes $H_\pm(z)$ (see Eq. (B.1) in the Appendix B) can be regarded as a partition function and $z \equiv \beta$ as the inverse temperature. Thus the related statistical free energy $\mathcal{F}(\beta)$, entropy S and internal energy E may be written respectively as

$$\mathcal{F}_p(\beta) = -\frac{1}{\beta} \log [H_-(\beta)], \quad (3.7)$$

$$\mathcal{F}_{sp}(\beta) = -\frac{1}{\beta} \log [H_+(\beta)H_-(\beta)], \quad (3.8)$$

$$E = \frac{\partial}{\partial \beta} [\beta \mathcal{F}(\beta)], \quad (3.9)$$

$$S = \beta^2 \frac{\partial}{\partial \beta} \mathcal{F}(\beta). \quad (3.10)$$

Using Eqs.(B.12) and (B.13) of the Theorem 1 one can obtain the asymptotic density of (super) p -brane states in the form (see also [39, 40, 41] for detail)

$$\Omega(M)dM \simeq 2C_{\pm}(p)M^{\frac{2p-q}{2(1+p)}} \exp \left[b_{\pm}(p)M^{\frac{2p}{1+p}} \right] dM, \quad (3.11)$$

$$b_{-}(p) \equiv \left(1 + \frac{1}{p} \right) [qA\Gamma(1+p)\zeta_R(1+p)]^{\frac{1}{1+p}}, \quad (3.12)$$

$$b_{+}(p) \equiv \left(1 + \frac{1}{p} \right) [qA\Gamma(1+p)(\zeta_{+}(1+p) + \zeta_{-}(1+p))]^{\frac{1}{1+p}}, \quad (3.13)$$

where according to Eq. (B.9) we have $A = V(S^{p-1})$. The asymptotic density is consistent with the entropy of near-extremal p -branes, indeed $S \sim M^{\frac{2p}{p+1}}$, i.e. the relation (3.6) holds. Note that asymptotic states density of compactified super p -branes which are protected against usual topological instabilities has the same form as above (and what is more the asymptotic behaviour of discrete states density has a universal character for all p -branes [41]).

In the limit $\beta \mapsto 0$ ($T \mapsto \infty$) the entropy of fundamental objects may be identified with $\log(r_{\pm}(N))$, while the internal energy is related to N . Hence from Eqs. (B.4),(B.5) and (3.7)-(3.10) one has

$$\mathcal{F}_p(T) \simeq -qA\Gamma(p)\zeta_R(1+p)T^{p+1}, \quad (3.14)$$

$$\mathcal{F}_{sp}(T) \simeq -qA\Gamma(p) [\zeta_{+}(1+p) + \zeta_{-}(1+p)] T^{p+1}, \quad (3.15)$$

$$E_p \simeq pqA\Gamma(p)\zeta_R(1+p)T^{p+1}, \quad (3.16)$$

$$E_{sp} \simeq pqA\Gamma(p) [\zeta_{+}(1+p) + \zeta_{-}(1+p)] T^{p+1}, \quad (3.17)$$

$$S_p \simeq (1+p)qA\Gamma(p)\zeta_R(1+p)T^p, \quad (3.18)$$

$$S_{sp} \simeq (1+p)qA\Gamma(p) [\zeta_{+}(1+p) + \zeta_{-}(1+p)] T^p. \quad (3.19)$$

Eliminating the quantity T between the Eqs. (3.16), (3.18) and (3.17), (3.19) one gets

$$S_p \simeq \frac{1+p}{p} [qpA\Gamma(p)\zeta_R(1+p)]^{\frac{1}{1+p}} E_p^{\frac{p}{1+p}}, \quad (3.20)$$

$$S_{sp} \simeq \frac{1+p}{p} [qpA\Gamma(p)(\zeta_{+}(1+p) + \zeta_{-}(1+p))]^{\frac{1}{1+p}} E_{sp}^{\frac{p}{1+p}}. \quad (3.21)$$

Thus the entropy behavior can be understood in terms of the degeneracy of some interacting fundamental p -brane excitation modes. Generally speaking the fundamental p -brane approach can yield a microscopic interpretation of the entropy.

It is well known that for $p > 1$ the free energy power series in Eqs. (C.20) and (C.21) (see Appendix C) diverge strongly for any β (supermembranes free energy at finite cut-off has been calculated in Refs. [42]). There is a conjecture that the critical temperature cannot be nonzero for $p > 1$. This argument has been suggested within the context of respective analysis of p -brane thermodynamics [43, 44]. On the same time the temperature goes to zero in the near-extremal limit for all p -brane solitons satisfying the condition (3.4) and such solitons have vanishing entropy in this limit as well.

More interesting possibility allows a finite temperature to be introduced into the quantized fundamental (super) p -brane theory [45]. The proof of this statement is presented in the Appendix C. Our calculation does not go through for even p , what suggests that there is some fundamental distinction between even and odd p in quantum theory of p -branes. Indeed the divergent serie in Eqs. (C.20) and (C.21) for the odd p -branes free energy, when reexpressed as ones on the left hand side of Eq. (C.28), remain well-defined for finite temperature and have a smooth $\beta \mapsto \infty$ ($T \mapsto 0$) limit. However it does prevent us from resumming the free energy,

Eqs. (C.20) and (C.21), for even p . Among other things, one can see a similar character of non-dilatonic solutions of black branes as well. The even p solutions are singular while the odd p solutions are not [46]. This may be related to the fact that for even p there cannot be a detailed agreement between the Bekenstein-Hawking and statistical entropy [14].

4 Near-Extremal Five-Branes and Black Hole Entropy

First let us suppose that the second quantized theory of five-brane can be considered as a theory of non-interacting strings. Then the Hilbert space of all multiple string states that satisfy the BPS conditions (zero branes) with a total energy momentum P has the form [13]

$$\mathcal{H}_P = \bigoplus_{\sum l N_l = N_P} \bigotimes_l \text{Sym}^{N_l} \mathcal{H}_l, \quad (4.1)$$

where symbol Sym^N indicates the N -th symmetric tensor product. One can expect that the exact dimension of \mathcal{H}_P is determined by the character expansion formula

$$\sum_{N_P} \dim \mathcal{H}_P q^{N_P} \simeq \prod_l \left(\frac{1+q^l}{1-q^l} \right)^{\frac{1}{2} \dim \mathcal{H}_l}, \quad (4.2)$$

where the dimension \mathcal{H}_l of the Hilbert space of single string BPS states with momentum $k = l\hat{P}$ is given by $\dim \mathcal{H}_l = d(\frac{1}{2}l^2\hat{P})$, and $|\hat{P}|^2 = |\hat{P}_L|^2 - |\hat{P}_R|^2$ (see Ref. [13] for detail). The asymptotics of the generating function (4.2) and the dimension \mathcal{H}_P can be found with the help of Theorem 1 (Appendix B) that is generalization of the Meinardus result for vector-valued functions.

The Eq. (4.2) is similar to the denominator formula of a (generalized) Kac-Moody algebra [47, 48]. A denominator formula can be written as follows

$$\sum_{\sigma \in W} (\text{sgn}(\sigma)) e^{\sigma(\rho)} = e^\rho \prod_{r>0} (1 - e^r)^{\text{mult}(r)}, \quad (4.3)$$

where ρ is the Weyl vector, the sum on the left hand side is over all elements of the Weyl group W , the product on the right side runs over all positive roots (one has the usual notation of root spaces, positive roots, simple roots and Weyl group, associated with Kac-Moody algebra) and each term is weighted by the root multiplicity $\text{mult}(r)$. For the $su(2)$ level, for example, an affine Lie algebra (4.3) is just the Jacobi triple product identity. For generalized Kac-Moody algebras there is a denominator formula

$$\sum_{\sigma \in W} (\text{sgn}(\sigma)) \sigma \left(e^\rho \sum_r \epsilon(r) e^r \right) = e^\rho \prod_{r>0} (1 - e^r)^{\text{mult}(r)}, \quad (4.4)$$

where the correction factor on the left hand side involves $\epsilon(r)$ which is $(-1)^n$ if r is the sum of n distinct pairwise orthogonal imaginary roots and zero otherwise.

The Eq. (4.2) reduces to the standard superstring partition function for $\hat{P}^2 = 0$ [13]. The equivalent description of the second quantized string states on the five-brane can be obtained by considering the sigma model on the target space $\sum_N \text{Sym}^N T^4$. There is the correspondence between the formula (4.1) and the term at order $q^{\frac{1}{2}N_P\hat{P}^2}$ in the expansion of the elliptic genus of the orbifold $\text{Sym}^{N_P} T^4$. Using this correspondence one finds that the asymptotic growth is equal that of states at level $\frac{1}{2}N_P\hat{P}^2$ in a unitary conformal field theory with central charge proportional to N_P .

In conclusion let us consider the D -brane method that may be used for calculation the ground state degeneracy of systems with quantum numbers of certain BPS extreme black holes. A typical 5-dimensional example has been analyzed in Refs. [3, 49, 50]. Working in the type IIB string

theory on $M^5 \otimes T^5$ one can construct a D -brane configuration such that the corresponding supergravity solutions describe 5-dimensional black holes. In this example five branes and one branes are wrapped on T^5 and the system is given Kaluza-Klein momentum N in one of the directions. Therefore the three independent charges (Q_1, Q_5, N) arise in the theory, where Q_1, Q_5 are electric and a magnetic charges respectively (see Eqs. (2.3) and (2.4)). The naive D -brane picture gives the entropy in terms of partition function $H_{\pm}(z)$ for a gas of $Q_1 Q_5$ species of massless quanta. For $p = 1$ the integers r_{\pm} in Eqs. (B.12) and (B.13) represent the degeneracy of the state with momentum N . Thus for $N \mapsto \infty$, using the Eqs. (B.12) and (B.13) of Theorem 1 one has

$$\log r_{\pm}(N) = \sqrt{q\zeta_{\pm}(2)}N - \frac{q+3}{4}\log N + \log C_{\pm}(1) + O(N^{-\kappa_{\pm}}), \quad (4.5)$$

where

$$C_{+}(1) = 2^{-\frac{1}{2}} \left(\frac{q}{16} \right)^{\frac{q+1}{2}}, \quad C_{-}(1) = C_{+}(1) \left(\frac{4}{3} \right)^{\frac{q+1}{2}}. \quad (4.6)$$

For fixed $q = 4Q_1 Q_5$ the entropy is given by

$$S_{sp} = \log [r_{+}(N)r_{-}(N)] \simeq 2\pi\sqrt{Q_1 Q_5 N} \left(\sqrt{\frac{2}{3}} + \sqrt{\frac{1}{2}} \right) - \left(\frac{3}{2} + 3Q_1 Q_5 \right) \log N. \quad (4.7)$$

This expression agrees with the classical black hole entropy.

Recently it has been pointed out that the classical result (4.7) is incorrect when the black hole becomes massive enough for its Schwarzschild radius to exceed any microscopic scale such as the compactification radii [49, 50]. Indeed, if the charges (Q_1, Q_5, N) tend to infinity in fixed proportion $Q_1 Q_5 = Q(N)$, then the correct formula does not agree with the black hole entropy (4.7). If, for example, $Q(N) = N$, then using Eqs. (B.12) and (B.13) for $N \mapsto \infty$ one finds $\log [r_{+}(N)r_{-}(N)] \sim N \log N$. The naive D-brane prescription, therefore, fails to agree with U-duality which requires symmetry among charges (Q_1, Q_5, N) [50].

5 Conclusions

In this paper we returned to the problem of asymptotic density of quantum states for fundamental p -branes initiated in Refs. [39, 45, 40]. We have shown that in the near-extremal regime the fundamental p -brane approach can yield a microscopic interpretation of the black hole entropy.

Indeed we realized a comparison between the asymptotic state density and the entropy (3.18), (3.19) of fundamental p -branes and classical black holes (3.3), (3.5). To this aim the explicit form of the total level brane density

$$\Omega(M) \simeq \mathcal{C}_{\pm}(p, M) \exp \left[b_{\pm}(p) M^{\frac{2p}{1+p}} \right], \quad (5.1)$$

where $\mathcal{C}_{\pm}(p, M) = 2C_{\pm}(p)M^{\frac{2p-q}{2(1+p)}}$, has been evaluated. Until our results the comparison between the statistical mechanical density of states of black holes and branes is based on some part of the density, the rest of it not being explicitly known. However, with the help of the Meinardus theorem (Eqs. (B.12) and (B.13) of the Appendix B) we have computed the complete p -brane state density (3.11), including the prefactors $\mathcal{C}_{\pm}(p, M)$ and the factors $b_{\pm}(p)$, depending on the dimension of the embedding space. A prefactor for the degeneracy of black hole states at mass level represents general quantum field corrections to the state density and it has not been known before our calculation. Nevertheless an attempt was made to compare the asymptotic state density of branes and related density of neutral black holes in Ref. [40]. In this paper we have shown that the asymptotic behaviour of classical entropy of near-extremal black branes coincides

with the asymptotic degeneracy of some weakly interacting fundamental p -brane excitation modes.

Further, using fundamental p -brane technology we have described and computed also the (near-extremal) entropy in the D -brane picture. The entropy is then just the sum of (left- and right-moving) contributions. We find that the remarkable Meinardus formulae can be used for the entropy calculation in a D -brane inspired picture. In particular, the explicit computation of the black hole (and massive black hole) entropy in terms of independent five-brane charges is given. At low energies and densities the statistical entropy (4.7) is in perfect agreement with results obtained in Ref. [12].

Thus we have evaluated the entropy by using classical black solutions and D -brane picture. A detailed correspondence of these approaches can be established with the help of fundamental p -brane partition function technique. The picture we advocated above relies rather on odd p . We hope that the methods used in the paper may shed light on the structure of fundamental extended objects at finite temperature and origin of p -branes entropy.

It has been demonstrated recently that BPS part of string spectrum for IIB string compactified on a circle do match with BPS part of supermembrane spectrum (see Refs. [51, 52]). In these papers the same discrete spectrum for supermembrane has been used as in the present work. This fact indicates that there are deep connections between strings and membranes (at least they should be considered as different corners of M-theory). Then different string results may be obtained via membrane-string correspondence. Therefore even being no fundamental theory the study of (super) p -branes may provide new deep insights in the understanding of string theory and consistent formulation of M-theory.

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6 Appendix A. (Super) p -Brane Mass Operator

It is known that for the noncompactified extended objects the question of reliability of the quasiclassical approximation is not absolutely clear [53, 54]. Nevertheless even for compactified p -branes the problem of the stability of classical solution is rather complicated and the loop diagrams have to be calculated to solve it.

Quasiclassical quantization of fundamental (super) p -branes which propagate in D -dimensional Minkowski space-time leads to the "number operators" $N_{\vec{n}}^{(b,f)}$, with $\vec{n} = (n_1, \dots, n_p) \in \mathbb{Z}^p$, where \mathbb{Z} is the ring of integer numbers. The operators $N_{\vec{n}}^{(b,f)}$ and the (anti) commutation relations for the oscillators (operators in a Fock space) can be found, for example, in Refs. [53, 54, 41]. The mass operators for the bosonic and supersymmetric p -branes can be written respectively as follows

$$M_p^2 = \sum_{i=1}^{D-p-1} \sum_{\vec{n} \in \mathbb{Z}^p / \{0\}} \omega_{\vec{n}} N_{\vec{n}i}^{(b)}, \quad (A.1)$$

$$M_{sp}^2 = \sum_{i=1}^{D-p-1} \sum_{\vec{n} \in \mathbb{Z}^p / \{0\}} \omega_{\vec{n}} \left(N_{\vec{n}i}^{(b)} + N_{\vec{n}i}^{(f)} \right), \quad (A.2)$$

where the frequencies are given by

$$\omega_{\vec{n}}^2 = \sum_{i=1}^p n_i^2. \quad (A.3)$$

7 Appendix B. Asymptotics for Generating Functions

Let us consider multi-component versions of the classical generating functions for partition functions, namely

$$H_{\pm}(z) = \prod_{\vec{n} \in \mathbb{Z}^p / \{0\}} [1 \pm \exp(-z\omega_{\vec{n}})]^{\pm q}, \quad (B.1)$$

where $z = y + 2\pi ix$, $\text{Re } z > 0$, $q > 0$ and $\omega_{\vec{n}}$ is given by Eq. (A.3). The total number of quantum states can be described by the quantities $r_{\pm}(N)$ defined by

$$K_{\pm}(t) = \sum_{N=0}^{\infty} r_{\pm}(N) t^N \equiv H_{\pm}(-\log t), \quad (B.2)$$

where $t < 1$, and N is a total quantum number. The Laurent inversion formula associated with the above definition takes the form

$$r_{\pm}(N) = \frac{1}{2\pi i} \oint dt \frac{K_{\pm}(t)}{t^{N+1}}, \quad (B.3)$$

where the contour integral is taken on a small circle about the origin.

We shall use the results of Meinardus [55, 56, 57] that can be easily generalised to the vector-valued functions of the (B.1) type (for more detail see Ref. [41]).

Proposition 1 *In the half-plane $\text{Re } z > 0$ there exists an asymptotic expansion for $H_{\pm}(z)$ uniformly in x as $y \mapsto 0$, provided $|\arg z| \leq \frac{\pi}{4}$ and $|x| \leq \frac{1}{2}$ and given by*

$$H_{+}(z) = \exp \left\{ q[A\Gamma(p)\zeta_{-}(1+p)z^{-p} - Z_p(0)\log 2 + O(y^{c_{+}})] \right\}, \quad (B.4)$$

$$H_{-}(z) = \exp \left\{ q[A\Gamma(p)\zeta_{+}(1+p)z^{-p} - Z_p(0)\log z + Z'_p(0) + O(y^{c_{-}})] \right\}, \quad (B.5)$$

where $0 < c_{+}, c_{-} < 1$ and $Z_p(s) \equiv Z_p\left|\frac{\vec{g}}{h}\right|(s)$ is the p -dimensional Epstein zeta function

$$Z_p\left|\frac{\vec{g}}{h}\right|(s) \equiv \sum'_{\vec{n} \in \mathbb{Z}^p} \left(\sum_{i=1}^p (n_i + g_i)^2 \right)^{-s/2} \exp[2\pi i(\vec{n}, \vec{h})], \quad (B.6)$$

which has a pole with residue A .

In above equations $\zeta_{-}(s) \equiv \zeta_R(s)$ is the Riemann zeta function, $\zeta_{+}(s) = (1 - 2^{1-s})\zeta_{-}(s)$, $(\vec{n}, \vec{h}) = \sum_{i=1}^p n_i h_i$, g_i and h_i are real numbers and the prime on \sum' means to omit the term $\vec{n} = -\vec{g}$. For $\text{Re } z < p$, $Z_p(s)$ is understood to be the analytic continuation of the right hand side of the Eq. (B.6). Furthermore $Z_p(s)$ is a fundamental zeta function which means it has a functional equation

$$Z_p\left|\frac{\vec{g}}{h}\right|(s) = \pi^{s-p/2} \frac{\Gamma(\frac{p-s}{2})}{\Gamma(\frac{s}{2})} \exp[-2\pi(\vec{g}, \vec{h})] Z_p\left|\frac{\vec{h}}{-\vec{g}}\right|(p-s). \quad (B.7)$$

Note that $Z_p(s)$ is an entire function in the complex s -plane except for the case when all the h_i are integers. In this case $Z_p(s)$ has a simple pole at $s = p$,

$$Z_p(p + \epsilon) = \frac{A}{\epsilon} B_p + O(\epsilon), \quad (B.8)$$

$$A = \frac{2\pi^{p/2}}{\Gamma(\frac{p}{2})}. \quad (B.9)$$

The constants B_p can be evaluated by means of a recursion formula. Finally from the functional equation (B.7) it follows that

$$Z'_p(0) = \frac{B_p}{A} + \frac{1}{2}\psi\left(\frac{p}{2}\right) - \log\pi - \frac{1}{2}\gamma, \quad (B.10)$$

$$Z_p(0) = -1, \quad (B.11)$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$ and γ is the Euler-Mascheroni constant.

By means of the asymptotic expansion of $K_{\pm}(t)$ for $t \mapsto 1$, which is equivalent to expansion of $H_{\pm}(z)$ for small z and using the formulae (B.4) and (B.5) one arrives at complete asymptotic of $r_{\pm}(N)$:

Theorem 1 *For $N \mapsto \infty$ one has*

$$r_{\pm}(N) = C_{\pm}(p) N^{(2qZ_p(0)-p-2)/(2(1+p))} \times \exp\left\{\frac{1+p}{p}[qA\Gamma(1+p)\zeta_{\pm}(1+p)]^{1/(1+p)} N^{p/(1+p)}\right\} [1 + O(N^{-\kappa_{\pm}})], \quad (B.12)$$

$$C_{\pm}(p) = [qA\Gamma(1+p)\zeta_{\pm}(1+p)]^{(1-2qZ_p(0))/(2p+2)} \frac{\exp(qZ'_p(0))}{[2\pi(1+p)]^{1/2}}, \quad (B.13)$$

$$\kappa_{\pm} = \frac{p}{1+p} \min\left(\frac{C_{\pm}}{p} - \frac{\delta}{4}, \frac{1}{2} - \delta\right),$$

and $0 < \delta < \frac{2}{3}$.

8 Appendix C. Free Energy of Fundamental (Super) p-Brane

The one-loop free energy of fields containing in (super) p -brane can be evaluated substituting the mass operators (A.1) and (A.2) and making use the Mellin-Barnes representation [58, 41]. Free energies have the form

$$\mathcal{F}_p(\beta) = -\frac{(4\pi)^{-\frac{D}{2}}}{4\pi i} \int_{Res=c} ds \zeta_{-}(s) \left(\frac{\beta}{2}\right)^{-s} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s-D}{2}\right) \text{Tr}[M_p^2]^{\frac{D-s}{2}}, \quad (C.1)$$

$$\mathcal{F}_{sp}(\beta) = -\frac{(4\pi)^{-\frac{D}{2}}}{2\pi i} \int_{Res=c'} ds \zeta_{+}(s) \left(\frac{\beta}{2}\right)^{-s} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s-D}{2}\right) \text{Tr}[M_{sp}^2]^{\frac{D-s}{2}}, \quad (C.2)$$

where

$$\text{Tr}[M_p^2]^{\frac{D-s}{2}} = \left[\Gamma\left(\frac{s-D}{2}\right)\right]^{-1} \int_0^{\infty} dt t^{\frac{s-D}{2}-1} H_{-}(t), \quad (C.3)$$

$$\text{Tr}[M_{sp}^2]^{\frac{D-s}{2}} = \left[\Gamma\left(\frac{s-D}{2}\right)\right]^{-1} \int_0^{\infty} dt t^{\frac{s-D}{2}-1} H_{+}(t) H_{-}(t), \quad (C.4)$$

and

$$H_{\pm}(t) = \prod_{\vec{n} \in \mathbb{Z}^p / \{0\}} [1 \pm \exp(-t\omega_{\vec{n}})]^{\pm(D-p-1)}. \quad (C.5)$$

Using Eqs.(B.4) and (B.5) of Proposition 1 one can obtain for $t \mapsto 0$,

$$H_{-}(t) = t^{\frac{q}{2}} \exp\left[V_{-}^{2p}(p)t^{-p} + U(p) + O(t^{c-})\right], \quad (C.6)$$

$$H_+(t)H_-(t) = \left(\frac{t}{2}\right)^{\frac{q}{2}} \exp \left[V_+^{2p}(p)t^{-p} + U(p) + O(t^{c_+}) \right], \quad (C.7)$$

where $q \equiv D - p - 1$ and

$$V_-(p) = \left[q 2^{p-2} \pi^{\frac{p-1}{2}} \Gamma\left(\frac{1+p}{2}\right) \zeta_-(1+p) \right]^{\frac{1}{2p}}, \quad (C.8)$$

$$V_+(p) = \left[q 2^{p-2} \pi^{\frac{p-1}{2}} \Gamma\left(\frac{1+p}{2}\right) (\zeta_-(1+p) + \zeta_+(1+p)) \right]^{\frac{1}{2p}}, \quad (C.9)$$

$$U(p) = \frac{q}{2} \left[\frac{\Gamma(\frac{p}{2})}{\pi p} B_p + \frac{1}{2} \psi\left(\frac{p}{2}\right) - \log \pi - \frac{1}{2} \gamma \right]. \quad (C.10)$$

It is convenient to use the same subtraction procedure as in Ref. [45] for the divergent term in H_{\pm} in order to regularize the free energy integrals,

$$\text{Tr}[M^2]^{\frac{D-s}{2}} = C_{\pm}(p) \frac{\Gamma\left(\frac{p-s+1}{2p}\right)}{\Gamma\left(\frac{s-D}{2}\right)} V_{\pm}^s(p) \text{Re}(-1)^{(s-p-1)/(2p)} + \frac{1}{\Gamma\left(\frac{s-D}{2}\right)} G_{\pm}(s; \mu), \quad (C.11)$$

where

$$C_-(p) = \frac{1}{p} \exp[U(p)] V_-^{-p-1}(p), \quad (C.12)$$

$$C_+(p) = \frac{1}{p} 2^{-\frac{q}{2}} \exp[U(p)] V_+^{-p-1}(p), \quad (C.13)$$

$$G_-(s; \mu) = \int_0^{\mu} dt t^{(s-D)/2-1} \left\{ H_-(t) - t^{q/2} \exp[V_-^{2p}(p)t^{-p} + U(p)] \right\}, \quad (C.14)$$

$$G_+(s; \mu) = \int_0^{\mu} dt t^{(s-D)/2} \left\{ H_+(t)H_-(t) - \left(\frac{t}{2}\right)^{q/2} \exp[V_+^{2p}(p)t^{-p} + U(p)] \right\}. \quad (C.15)$$

In Eqs. (C.14) and (C.15) an infrared cutoff parameter μ has been introduced. It should be noted that in contrast with the supersymmetric case the regularization in the infrared region of the integral defining the analytic function $G_-(s; \mu)$ cannot be removed (there are tachyons in the p -brane spectrum).

The one-loop free energy can be written as follows

$$\mathcal{F}_p(\beta) = -\frac{(4\pi)^{-\frac{D}{2}}}{4\pi i} \int_{\text{Re } s=c} ds [\Phi_-(s) + \Xi_-(s)], \quad (C.16)$$

$$\mathcal{F}_{sp}(\beta) = -\frac{(4\pi)^{-\frac{D}{2}}}{2\pi i} \int_{\text{Re } s=c'} ds [\Phi_+(s) + \Xi_+(s)], \quad (C.17)$$

where

$$\Phi_{\pm}(s) = C_{\pm} x_{\pm}^s \text{Re}(-1)^{(s-p-1)/(2p)} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{p+1-s}{2p}\right) \zeta_{\pm}(s), \quad (C.18)$$

$$\Xi_{\pm}(s) = G_{\pm}(s; \mu) \left(\frac{\beta}{2}\right)^{-s} \Gamma\left(\frac{s}{2}\right) \zeta_{\pm}(s), \quad (C.19)$$

and $x_{\pm} = 2V_{\pm}(p)\beta^{-1}$. The various residues of the meromorphic functions $\Phi_{\pm}(s)$ and $\Xi_{\pm}(s)$ are

$$\text{Res}[\Phi_{\pm}(s), s = p(2k-1)+1] = C_{\pm}(p) \frac{\Gamma\left(pk + \frac{1-p}{2}\right)}{\Gamma(k)} \zeta_{\pm}(2pk+1-p) x_{\pm}^{1+p(2k-1)},$$

$$\text{Res} [\Phi_-(s), s = 0] = -C_-(p) \Gamma \left(\frac{1+p}{2p} \right) \text{Re}(-1)^{-(1+p)/(2p)},$$

$$\text{Res} [\Xi_\pm(s), s = 1] = \sqrt{\pi} 2^{\mp 1} G_\pm(1; \mu) \beta^{-1},$$

$$\text{Res} [\Xi_-(s), s = 0] = -G_-(0; \mu).$$

Finally a formal computation of the integrals (C.16) and (C.17) gives

$$\begin{aligned} \mathcal{F}_p(\beta) = & -\frac{1}{2(4\pi)^{D/2}} \left\{ C_-(p) \sum_{k=1}^{\infty} \frac{\Gamma \left(pk + \frac{1-p}{2} \right)}{\Gamma(k)} \zeta_R(2pk + 1 - p) x_-^{1+p(2k-1)} + \right. \\ & \left. 2\sqrt{\pi} G_-(1; \mu) \beta^{-1} - G_-(0; \mu) - C_-(p) \Gamma \left(\frac{1+p}{2p} \right) \text{Re}(-1)^{-(1+p)/(2p)} \right\} + \\ & \mathcal{F}_{p\mathcal{R}}(x) + \mathcal{F}_{p\mathcal{R}}(x; \mu), \end{aligned} \quad (C.20)$$

$$\begin{aligned} \mathcal{F}_{sp}(\beta) = & -\frac{1}{(4\pi)^{D/2}} \left\{ C_+(p) \sum_{k=1}^{\infty} \frac{\Gamma \left(pk + \frac{1-p}{2} \right)}{\Gamma(k)} \zeta_+(2pk + 1 - p) x_+^{1+p(2k-1)} + \right. \\ & \left. \frac{1}{2} \sqrt{\pi} G_+(1; \mu) \beta^{-1} \right\} + \mathcal{F}_{sp\mathcal{R}}(x) + \mathcal{F}_{sp\mathcal{R}}(x; \mu), \end{aligned} \quad (C.21)$$

where $\mathcal{F}_{p\mathcal{R}}(x)$, $\mathcal{F}_{sp\mathcal{R}}(x)$ and $\mathcal{F}_{p\mathcal{R}}(x; \mu)$, $\mathcal{F}_{sp\mathcal{R}}(x; \mu)$ are the contributions $\Phi_\pm(s)$ and $\Xi_\pm(s)$ along the arc of radius \mathcal{R} in the right half-plane. If $|x| < 1$ then $\mathcal{F}_{p\mathcal{R}}(x)$ and $\mathcal{F}_{sp\mathcal{R}}(x)$ vanishes when $\mathcal{R} \mapsto \infty$.

Strings

Returning to the string case we note that for $p = 1$ the serie in Eqs. (C.20) and (C.21) can be evaluated in the closed forms [41], [?]. The sums of these serie for open string and superstring (without gauge group) have respectively the forms

$$\sum_{k=1}^{\infty} \zeta_R(2k) x_-^{2k} = \frac{1}{2} - \frac{1}{2} \cot(\pi x), \quad (C.22)$$

$$\sum_{k=1}^{\infty} \zeta_R(2k) (1 - 2^{-2k}) x_+^{2k} = \frac{\pi x}{4} \tan \left(\frac{\pi x}{2} \right). \quad (C.23)$$

The finite radius of Laurent series convergence $|x_\pm| < 1$ corresponds to the critical temperature in string thermodynamics: $x_\pm = \beta_\pm^c / \beta$ and $\beta_\pm^c = 2V_\pm(1)$. As a result $\beta_-^c = \sqrt{8}\pi$ and $\beta_+^c = 2\pi$ (review of string theory at non-zero temperature may be found in Ref. [59]). Written in trigonometric form Expressions (C.22) and (C.23) displays a certain periodicity in temperature, the physical meaning of which is still obscure. For the both open bosonic and supersymmetric strings the β -behavior of the free energy (C.22), (C.23) has the dependence on temperature near the Hagedorn transition which looks like one found in Ref. [60].

p-Branes

For $p > 0$ the power series (C.20) and (C.21) are divergent for any $x_\pm > 0$. Nevertheless one can construct an analytic continuation of these expressions. Let us define for $|z| < \infty$ two serie

$$\mathcal{J}_\pm(z) = \sum_{k=0}^{\infty} \frac{\sqrt{\pi}}{\Gamma(k+1) \Gamma \left(pk + \frac{p+2}{2} \right)} \nu_\pm(k; p) \left(\frac{z}{2} \right)^{p(2k+1)+1}. \quad (C.24)$$

In addition the factors $\nu_\pm(k; p)$ have the form

$$\nu_-(k; p) = (-1)^{pk+1}, \quad (C.25)$$

$$\nu_+(k; p) = \nu_-(k; p) \left[1 - 2^{-p(2k+1)-1} \right]. \quad (C.26)$$

For finite variable z these series converge and convergence is improving rapidly with increasing integer number p . Let $z_{\pm} = j \cdot 2\pi x_{\pm}$, then for the series

$$\sum_{j=1}^{\infty} \mathcal{J}_{\pm}(j \cdot 2\pi x_{\pm}) = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \frac{\sqrt{\pi}}{\Gamma(k+1)\Gamma\left(pk + \frac{p+2}{2}\right)} \nu_{\pm}(k; p) (j\pi x_{\pm})^{p(2k+1)+1}, \quad (C.27)$$

one can commute the (now divergent) sum Σ_j through Σ_k which generates extra terms on the right $J_{\pm}(x, p)$. Thus the result is

$$\begin{aligned} & \sum_{j=1}^{\infty} \mathcal{J}_{\pm}(j2\pi x_{\pm}) + J_{\pm}(x, p) \\ &= \sum_{k=0}^{\infty} \frac{\sqrt{\pi}}{\Gamma(k+1)\Gamma\left(pk + \frac{p+2}{2}\right)} \nu_{\pm}(k; p) \zeta_R[-p(2k+1)] (\pi x_{\pm})^{p(2k+1)+1} \\ &= \sin\left(\frac{\pi p}{2}\right) \sum_{k=1}^{\infty} \frac{\Gamma\left(pk + \frac{1-p}{2}\right)}{\Gamma(k)} \zeta_{\pm}(2pk+1-p) x_{\pm}^{p(2k-1)+1}, \end{aligned} \quad (C.28)$$

where, for example, $J_-(x, p) = \pi \left[px \Gamma\left(\frac{(p-1)(p+1)}{2p}\right) \right]^{-1}$ (see Ref. [45]), and in the second equality the functional equation for $\zeta_R(s)$ is used. The vanishing of the factor $\sin(p\pi/2)$ on the right hand side for even p merely expresses the vanishing of $\zeta_R[-p(2k+1)]$ in the first equality. For $p = 2k, k \in \mathbb{Z}_+$, the right hand side of Eq. (C.28) is zero and the function (C.27) therefore sums to a single term.

References

- [1] A. Strominger and C. Vafa, Phys. Lett. **B 379**, 99 (1996).
- [2] S.R. Das and S. Mathur, "*Excitations of D-Strings, Entropy and Duality*", hep-th/9601152 (1996).
- [3] C. Callan and J. Maldacena, Nucl. Phys. **B 475**, 645 (1996).
- [4] G.T. Horowitz and A. Strominger, Phys. Rev. Lett. **77**, 2368 (1996).
- [5] A. Ghosh and P. Mitra, "*Entropy of Extremal Dyonic Black Holes*", hep-th/9602057 (1996).
- [6] J. Beckenridge, R. Meyers, A. Peet and C. Vafa, "*D-Branes and Spinning Black Holes*", hep-th/9602065 (1996).
- [7] S.S. Gubser, I.R. Klebanov and A.W. Peet, "*Entropy and Temperature of Black 3-Branes*", hep-th/9602135 (1996).
- [8] S.R. Das, "*Black Hole Entropy and String Theory*", hep-th/9602172 (1996).
- [9] J. Beckenridge, D.A. Lowe, R. Myers, A. Peet, A. Strominger and C. Vafa, Phys. Lett. **B 381**, 423 (1996).
- [10] J. Maldacena and A. Strominger, Phys. Rev. Lett. **77**, 428 (1996).
- [11] C.V. Johnson, R.R. Khuri and R.C. Myers, Phys. Lett. **B 378**, 78 (1996).

- [12] G.T. Horowitz, J. Maldacena and A. Strominger, Phys. Lett. **B 383**, 151 (1996).
- [13] R. Dijkgraaf, E. Verlinde and H. Verlinde, "*BPS Spectrum of the Five- Brane and Black Hole Entropy*", hep-th/9603126 (1996).
- [14] I.R. Klebanov and A.A.Tseytlin, "*Entropy of Near-Extremal Black p-Branes*", hep-th/9604089 (1996).
- [15] H. Lü, S. Mukherji, C.N. Pope and J. Rahmfeld, "*Loop-Corrected Entropy of Near-Extremal Dilatonic p-Branes*", hep-th/9604127 (1996).
- [16] M.J. Duff and K.S. Stelle, Phys. Lett. **B 253**, 113 (1991).
- [17] R. Güven, Phys. Lett. **B 276**, 49 (1992).
- [18] G.T. Horowitz and A. Strominger, Nucl. Phys. **B 360**, 197 (1991).
- [19] M.J. Duff and J.X. Lu, Phys. Lett. **B 273**, 409 (1991).
- [20] M.J. Duff, S. Ferrara, R.R. Khuri and J. Rahmfeld, Phys. Lett. **B273**, 479 (1995).
- [21] H. Lü and C.N. Pope, "*Multi-Scalar p-Brane Solitons*", hep-th/9512153 (1995).
- [22] E. Halyo, "*Reissner-Nordström Black Holes and Strings with Rescaled Tension*", hep-th/9610068 (1996).
- [23] H. Lü, C.N. Pope, E. Sezgin and K.S. Stelle, Nucl. Phys. **B 456**, 669 (1995).
- [24] M. Cvetič and D. Youm, Phys. Rev. **B 53**, 584 (1996).
- [25] M.J. Duff and J. Rahmfeld, Phys. Lett. **B 345**, 441 (1995).
- [26] A. Sen, Nucl. Phys. **B 440**, 421 (1995).
- [27] A. Sen, Mod. Phys. Lett. **A 10**, 2081 (1995).
- [28] M.J. Duff, J.T. Liu and J. Rahmfeld, Nucl. Phys. **B 459**, 125 (1996).
- [29] J. Rahmfeld, "*Extremal Black Holes as Bound States*", hep-th/9512089 (1995).
- [30] H. Lü and C.N. Pope, Nucl. Phys. **B 465**, 127 (1996).
- [31] H. Lü and C.N. Pope, "*Black p-Branes and Their Vertical Dimensional Reduction*", hep-th/9609126 (1996).
- [32] M.J. Duff, H. Lü and C.N. Pope, "*The Black p-Branes of M-Theory*", hep-th/9604052 (1996).
- [33] H. Lü, S. Mukherji and C.N. Pope, "*Cosmological Solutions in String Theories*", hep-th/9610107 (1996).
- [34] A. Lukas, B.A. Ovrut and D. Waldram, "*String and M-Theory Cosmological Solutions with Ramond Forms*", hep-th/9610238 (1996).
- [35] A. Dabholkar, G.W. Gibbons, J.A. Harvey and F. Ruiz Ruiz, Nucl. Phys. **B 340**, 33 (1990).
- [36] G.G. Callan, J.A. Harvey and A. Strominger, Nucl. Phys. **B 359**, 611 (1991).
- [37] G.G. Callan, J.A. Harvey and A. Strominger, Nucl. Phys. **B 367**, 60 (1991).

- [38] M.J. Duff and J.X. Lu, Nucl. Phys. **B 416**, 301 (1994).
- [39] A.A. Bytsenko, K. Kirsten and S. Zerbini, Phys. Lett. **B 304**, 235 (1993).
- [40] A.A. Bytsenko, K. Kirsten and S. Zerbini, Mod. Phys. Lett. **A 9**, 1569 (1994).
- [41] A.A. Bytsenko, G. Cognola, L. Vanzo and S. Zerbini, Phys. Reports **266**, 1 (1996).
- [42] A.A. Bytsenko and S.D. Odintsov, Phys. Lett. **B 243**, 63 (1990); **B 245**, 21 (1990).
- [43] E. Alvarez, T. Ortin and M.A.R. Osorio, Phys. Rev. **D 43**, 3990 (1991).
- [44] E. Alvarez and T. Ortin, Mod. Phys. Lett. **A 7**, 2889 (1992).
- [45] A.A. Actor and A.A. Bytsenko, Phys. Lett. **B 315**, 74 (1993).
- [46] G.W. Gibbons, G.T. Horowitz and P.K. Townsend, Class. Quant. Grav. **12**, 297 (1995).
- [47] R.E. Borcherds, Invent. Math. **120**, 161 (1995).
- [48] J.A. Harvey and G. Moore, Nucl. Phys. **B 463**, 315 (1996).
- [49] J. Maldacena and L. Susskind, Nucl. Phys. **B 475**, 679 (1996).
- [50] E. Halyo, A. Rajaraman and L. Susskind, "*Branless Black Holes*", hep-th/9605112 (1996).
- [51] J.H.Schwarz, Phys. Lett. **B 360**, 13 (1995) (E: **B 364**, 252 (1995)).
- [52] J.G. Russo and A.A. Tseytlin, "*Waves, Boosted Branes and BPS States in M-Theory*", hep-th 9611047 (1996).
- [53] E. Bergshoeff, E. Sezgin and P.K. Townsend, Ann. Phys. **185**, 330 (1987).
- [54] M.J. Duff, T. Inami, C.N. Pope, E. Sezgin and K. Stelle, Nucl. Phys. **B 297**, 515 (1988).
- [55] G. Meinardus, Math. Z. **59**, 338 (1954).
- [56] G. Meinardus, Math. Z. **61**, 289 (1954).
- [57] G.E. Andrews, "*The Theory of Partitions*". In Encyclopedia of Mathematics and its Applications, Addison-Wesley Publishing Company (1976).
- [58] A.A. Bytsenko, E. Elizalde, S.D. Odintsov and S. Zerbini, Nucl. Phys. **B 394**, 423 (1993).
- [59] S.D. Odintsov, Rivista Nuovo Cim. **15**, 1 (1992).
- [60] J.J. Atick and E. Witten, Nucl. Phys. **B 310**, 291 (1988).
- [61] T. Muto, Phys. Lett. **B 391**, 310 (1997).